B. Math First Semester 2005 Final Examination
Analysis 1 24-11-05

Answer all the questions. All answers require justification.
If you are using a theorem/result proved in the class, state it correctly.
Points: 6 x 10 = 60. Time: 3hrs

1) Define a countable set and show that the union of a sequence of countable sets is countable.

2) Let $\{r_n\}_{n\geq 1}$ be an enumeration of rational numbers in $[0,1]$. Show that there exists a nested sequence of closed intervals $[a_n, b_n] \subset [0,1]$ such that $r_n \notin [a_n, b_n]$ for all n.

3) Compute the following limits. Give reasons for your conclusion.
a) $\lim_{n\to\infty} \frac{1}{2^n}(1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!})$
b) Let $\{x_n\}_{n\geq 1} \subset (-1,1)$. Suppose $\lim_{n\to\infty} \frac{x_n - 1}{x_n + 1} = 0$. Compute $\lim_{n\to\infty} x_n$.

4) a) Let $\sum a_n$ be an absolutely convergent series. Show that every rearrangement of the series $\sum a_n$ converges.
b) Show that $\sum (\frac{\sin n}{n})^{\frac{1}{n}}$ converges.

5) Let $f : [0,1] \to [0,1]$ be a continuous function. Show that for any positive integer n, there exists a $x_0 \in [0,1]$ such that $\sup\{\frac{x^n}{1+f(x)} : x \in [0,1]\} = \frac{x_0^n}{1+f(x_0)}$.

6) Let $g : [-1,1] \to (0,1)$ be a continuous function. Show that g is not an onto map.

7) Let **N** denote the set of natural numbers. Let $g : R \to R$ be defined by $g(x) = \inf\{|x - n| : n \in \mathbf{N}\}$.
Show that g is uniformly continuous.

8) Let $f : (0,1) \to R$ be a function. Show that f is differentiable at a point $c \in (0,1)$ if and only if there exists a function $A : (0,1) \to R$ that is continuous at c such that $f(x) - f(c) = A(x)(x - c)$ for all $x \in (0,1)$.

9) Let $f : R \to R$ be a differentiable function. Suppose for every x, $f'(x) \neq 0$. Show that f is an injection and $f(R)$ is an open interval.

10) Let $f : [-1,1] \to R$ be a thrice differentiable function. Suppose $f(-1) = 0 = f(0) = f'(0)$ and $f(1) = 1$.
Show that there exists $s \in (0,1)$, $t \in (-1,0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.